A proof of the E_0 dichotomy from the G_0 dichotomy

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Let E be a Borel equivalence relation on a Polish space X. Let $Y = {\sim}E = X^2 \setminus E$, and put a Polish topology on Y making the projections continuous. Define a *directed* Borel graph G on Y by setting $(a_0, b_0) G (a_1, b_1) \iff a_0 E b_1$.

Claim 1. E is smooth iff $\chi_B(G) \leq \aleph_0$.

Proof. Recall that E is smooth iff it has a countable Borel separating family, i.e., a sequence of Borel sets $B_i \subseteq X$ so that $aEb \iff \forall i (a \in B_i \iff b \in B_i)$.

Suppose E is smooth, and let B_i be a countable Borel separating family for E. Then $Y = \bigcup_i (B_i \times {\sim}B_i) \cup ({\sim}B_i \times B_i)$, and these sets are G-independent.

Conversely, suppose that A_i is a cover of Y by countably many G-independent Borel sets. Then the pairs $(\mathrm{proj}_0(A_i), \mathrm{proj}_1(A_i))$ are E-independent, i.e., if $a \in \mathrm{proj}_0(A_i), b \in \mathrm{proj}_1(A_i)$ then $a \not E b$. By a standard reflection argument, there is a Borel E-invariant set $B_i \supseteq \mathrm{proj}_0(A_i)$ so that $(B_i, \mathrm{proj}_1(A_i))$ is E-independent, and it is easy to verify that the sets B_i form a countable Borel separating family for E.

Remark 2. Ben Miller has pointed out that the graph G we define here was first considered (at least in the context of descriptive set theory) by Louveau, and Claim 1 was essentially proven by Lecomte (see [LM08], bottom of page 2).

Suppose now that E is not smooth. Let $S \subseteq 2^{<\omega}$ be a dense set which contains exactly one sequence of every finite length $n \in \omega$, and recall that G_0 is the *directed* Borel graph on 2^ω defined by

$$G_0 = \{(s^\frown(0)^\frown x, s^\frown(1)^\frown x) : s \in S, x \in 2^\omega\},$$

where $^\frown$ denotes concatenation of sequences. By the directed G_0 dichotomy and the previous claim, there is a continuous homomorphism $\psi : 2^\omega \to Y$ from G_0 to G.

Let now H be the (undirected) graph given by

$$xHz \iff \exists y (x G_0 y \ \& \ z G_0 y),$$

and define xFy iff x, y are in the same connected component of H. Let E' be the pullback of E along $\mathrm{proj}_0 \circ \psi$, and let D be the pullback of $\Delta(X)$ along $\mathrm{proj}_0 \circ \psi$.

Claim 3. E' is meagre and D is closed and nowhere dense.

Proof. As $\mathrm{proj}_0 \circ \psi$ is continuous and $D \subseteq E'$, it suffices to show that E' is meagre. By the Kuratowski–Ulam Theorem, it suffices to show that every E'-class is meagre. Let C be an E'-class, and suppose for the sake of contradiction that C is non-meagre. Then by a standard argument, there are $x, y \in C$ with xG_0y. If we write $\psi(x) = (a_0, b_0), \psi(y) = (a_1, b_1)$, then we have a_0Ea_1 (as x, y are in the same E'-class) and a_0Eb_1 (as ψ is a directed homomorphism from G_0 to G). It follows that a_1Eb_1 and $\psi(y) = (a_1, b_1) \in Y = {\sim}E$, a contradiction.

Claim 4. Let $R \subseteq 2^\omega \times 2^\omega$ be meagre and $D \subseteq 2^\omega \times 2^\omega$ be closed and nowhere dense. Then there is a continuous homomorphism $\phi : (G_0, {\sim}\Delta(2^\omega), {\sim}E_0) \to (F, {\sim}D, {\sim}R)$.

Apply these two claims to E', D, in order to get a continuous homomorphism $\phi : (G_0, {\sim}\Delta(2^\omega), {\sim}E_0) \to (F, {\sim}D, {\sim}E')$. Then $\mathrm{proj}_0 \circ \psi \circ \phi : 2^\omega \to X$ is a continuous embedding of E_0 into E. Indeed, by definition of D this composition is injective. Moreover, $\mathrm{proj}_0 \circ \psi$ is a reduction of E' into E, and ϕ is a reduction of E_0 into E'. (To see this last part, note that $H \subseteq E'$, and thus $F \subseteq E'$: if xHz, then there is some y so that xG_0y and zG_0y, in which case $\mathrm{proj}_0(\psi(x)) E \,\mathrm{proj}_1(\psi(y)) E \,\mathrm{proj}_0(\psi(z))$.)

Thus it remains only to prove Claim 4.

Proof of Claim 4. We define directed graphs G_0 on 2^n by setting, for $u, v \in 2^n$, uG_0v iff there are $s \in S \cap 2^{<n}, t \in 2^{n-1-|s|}$ so that $u = s^\frown(0)^\frown t, v = s^\frown(1)^\frown t$. We define (undirected) graphs H on 2^n by $uHv \iff \exists w(uG_0w \ \& \ vG_0w)$, and let F be the equivalence on 2^n given by the connected components of H. (We abuse notation here and let G_0, H, F denote graphs on 2^n for all $n \leq \omega$.) Note that

$$\forall \star \in \{G_0, H, F\}, u, v \in 2^n, x \in 2^{\leq\omega}(u \star v \iff u^\frown x \star v^\frown x). \qquad (*)$$

Subclaim 5. For any $u, v \in 2^n$, there are some $u', v' \in 2^k, k < \omega$ so that $u^\frown u' F v^\frown v'$.

Proof. It is a standard fact that for all $u, v \in 2^n, n < \omega$ there is an undirected path from u to v in G_0. (One can prove this easily by induction on n using $(*)$.) We prove the claim for all $u, v \in 2^n, n < \omega$, by induction on the distance from u to v in G_0.

This is clearly true if the distance is 0, i.e., $u = v$. Suppose now that u, v are neighbours, i.e., uG_0v or vG_0u. By the symmetry of F, we may assume wlog that uG_0v. As S is dense, there is some $t \in 2^{<\omega}$ with $v^\frown t \in S$. But then $u^\frown t^\frown 1 G_0 v^\frown t^\frown 1$ by $(*)$, and $v^\frown t^\frown 0 G_0 v^\frown t^\frown 1$, so that $u' = t^\frown 1, v' = t^\frown 0$ works.

Finally, suppose that we have shown that this holds for all u, v of distance at most m in G_0, for $m \geq 1$, and let u, v be of distance $m+1$ apart in G_0. Let w be the last vertex before v on an undirected path from u to v of length $m+1$. By assumption, there are $u', v' \in 2^k$ with $u^\frown u' F w^\frown v'$. Note that $w^\frown v', v^\frown v'$ are adjacent in G_0 by $(*)$, so by assumption there are $u'', v'' \in 2^l$ with $w^\frown v'^\frown u'' F v^\frown v'^\frown v''$. By $(*)$ and the transitivity of F,

$$u^\frown u'^\frown u'' F w^\frown v'^\frown u'' F v^\frown v'^\frown v'',$$

so the pair $u'^\frown u'', v'^\frown v'' \in 2^{k+l}$ works for u, v.

Let now U_n be a decreasing sequence of dense open sets in $2^\omega \times 2^\omega$ so that $D \cap U_0 = \emptyset$ and $R \cap \bigcap_n U_n = \emptyset$. We will define recursively pairs of finite sequences $(t_0^n, t_1^n) \in \bigcup_{m>0} 2^m \times 2^m$ and maps $\phi_n : 2^n \to 2^{<\omega}$ satisfying for all $n < \omega$:

1. $\phi_0(\emptyset) = \emptyset$;
2. $\phi_{n+1}(u^\frown(i)) = \phi_n(u)^\frown t_i^n$ for $u \in 2^n, i \in 2$;
3. $N_{\phi_n(u)^\frown t_0^n} \times N_{\phi_n(v)^\frown t_1^n} \subseteq U_n$ for all $u, v \in 2^n$; and
4. if $s \in S \cap 2^n$, then $\phi_n(s)^\frown t_0^n F \phi_n(s)^\frown t_1^n$.

(Here, $N_t = \{x \in 2^\omega : t \subseteq x\}$ for $t \in 2^{<\omega}$.)

Assuming this has been done, define $\phi(x) = \bigcup_n \phi_n(x \restriction n)$ for $x \in 2^\omega$. Then $\phi : 2^\omega \to 2^\omega$ is continuous, and by 2, 3 it is a homomorphism $\sim\Delta(2^\omega) \to \sim D$ and $\sim E_0 \to \sim R$. It is also easy to see by 2 that for any $u, v \in 2^n, x \in 2^\omega$, there is some $y \in 2^\omega$ so that $\phi(u^\frown x) = \phi_n(u)^\frown y, \phi(v^\frown x) = \phi_n(v)^\frown y$. By 2, 4 and $(*)$, it follows that ϕ is a homomorphism from G_0 to F.

It remains to define the sequences (t_0^n, t_1^n) and maps ϕ_n. Suppose we have constructed these sequences for all $m < n$. We define ϕ_n to be the unique map satisfying 1, 2. Because U_n is dense and open, there are sequences w_0, w_1 so that $N_{\phi_n(u)^\frown w_0} \times N_{\phi_n(v)^\frown w_1} \subseteq U_n$ for all $u, v \in 2^n$. (That this is true for a single pair u, v is immediate; we recursively apply this fact to all such pairs, extending w_0, w_1 at each step, in order to find sequences that work for all u, v.) We may ensure that $|w_0| = |w_1| > 0$.

Let now $s \in S \cap 2^n$. By the subclaim, there are $w_0', w_1' \in 2^k, k < \omega$ so that $\phi_n(s)^\frown w_0^\frown w_0' F \phi_n(s)^\frown w_1^\frown w_1'$. We let $t_i^n = w_i^\frown w_i'$, and note that 3, 4 are satisfied by this choice.

[LM08] Dominique Lecomte and Benjamin D. Miller, Basis theorems for non-potentially closed sets and graphs of uncountable Borel chromatic number, *J. Math. Log.* **8(2)** (2008), 121–162, DOI: 10.1142/S0219061308000749.